Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Topics in Discrete Probability

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Max. points : 20. **Time Limit : $2\frac{1}{2}$ hours.**

Answer any two questions only.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class or assignments, mention it clearly.

1. Let $C(G)$ be the size of the largest clique (i.e., a complete subgraph) in a graph G. Let $G(n, P)$ denote the Erdös-Rényi graph with n vertices and the probability of an edge being present is p. Show the following.
   (a) Let $n^2 p_n^3 \to 0$. Show that $\mathbb{P}(C(G(n,p)) \geq 4) \to 0$. (**3**).
   (b) Let $n^2 p_n^3 \to \infty$. Show that $\mathbb{P}(C(G(n,p)) \geq 4) \to 1$. (**7**).

2. (a) For $d, k \in \mathbb{N}$. Consider the graph $G_{d,k} = \mathbb{Z}^d \times \{1, \ldots, k\}$. For what values of d and k does the bond percolation on $G_{d,k}$ have a non-trivial phase transition i.e., $p_c(G_{d,k}) \in (0, 1)$. (**3**).
   (b) Let $d \geq 1$ and $e_1 = (1, 0, \ldots, 0)$ denote the standard unit vector in $\mathbb{R}^d$. Consider the bond percolation on $\mathbb{Z}^d$. Define $\tau_p(x, y) := \mathbb{P}_p(x \leftrightarrow y)$ for $x, y \in \mathbb{Z}^d$. Show the following :
      i. Show that $\eta(p) := \lim_{n\to\infty} \frac{-\log \tau_p(O, ne_1)}{n}$ exists and is finite for all $p > 0$. (**3**)
      ii. Show that $\eta(p)$ is non-increasing for all p, strictly positive for all $p \in (0, p_c)$ and 0 for all $p > p_c$. (**4**)
      Hint : Compare $\tau_p(O, 2ne_1)$ and $\tau_p(O, x)^2$ for any $x \in \partial\Lambda_n$.

3. (a) Let $d \geq 1$ and consider bond percolation on $\mathbb{Z}^d$. Show that for all $x, y \in \mathbb{Z}^d$,

$$\tau_p(x, y) = \lim_{n\to\infty} \mathbb{P}_p\left(\{x \overset{\Lambda_n}{\leftrightarrow} y\} \cup (\{x \leftrightarrow \partial\Lambda_n\} \cap \{y \leftrightarrow \partial\Lambda_n\})\right)$$

and that $\tau_p(x,y)$ is continous as a function of p. (**5**)

(b) Consider bond percolation on $\mathbb{Z}^2$ with $p = 1/2$. Call G the subgraph induced by bond percolation on $\mathbb{Z}^2$ and G^* the dual subgraph on the dual lattice $(\mathbb{Z}^*)^2$. Let $R_n := ([-(n-1), n] \times [-n, n]) \cap \mathbb{Z}^2$ be a rectangle on the planar square lattice and $R_n^* := \big((\frac{1}{2}, \frac{1}{2}) + [-n, n] \times [-n, n-1]\big) \cap (\mathbb{Z}^*)^2$ be the rotation of R_n but in the dual lattice. Let B_n be the event that there exist infinite G-paths from the top and bottom of ∂R_n in $\mathbb{Z}^2 \setminus R_n$ and infinite G^*-paths from the left and right of ∂R_n^* in $(\mathbb{Z}^*)^2 \setminus R_n^*$. Show that

$$\mathbb{P}_{1/2}[B_n] \geq 1 - 4\mathbb{P}_{1/2}[\partial R_n \nleftrightarrow \infty]^{1/4}$$

and using the above inequality argue that $p_c(\mathbb{Z}^2) \geq 1/2$. (**5**)